

Exercise 79

If f and g are both even functions, is $f + g$ even? If f and g are both odd functions, is $f + g$ odd? What if f is even and g is odd? Justify your answers.

Solution

Suppose that f and g are even: $f(-x) = f(x)$ and $g(-x) = g(x)$. Consider the sum of f and g .

$$(f + g)(x) = f(x) + g(x)$$

Substitute $-x$ for x .

$$\begin{aligned}(f + g)(-x) &= f(-x) + g(-x) \\ &= f(x) + g(x) \\ &= (f + g)(x)\end{aligned}$$

Therefore, the sum of f and g is even as well. Suppose that f and g are odd: $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Consider the sum of f and g and substitute $-x$ for x .

$$\begin{aligned}(f + g)(-x) &= f(-x) + g(-x) \\ &= -f(x) - g(x) \\ &= -(f(x) + g(x)) \\ &= -(f + g)(x)\end{aligned}$$

Therefore, the sum of f and g is odd as well. Suppose that f is even and g is odd: $f(-x) = f(x)$ and $g(-x) = -g(x)$. Consider the sum of f and g and substitute $-x$ for x .

$$\begin{aligned}(f + g)(-x) &= f(-x) + g(-x) \\ &= f(x) - g(x)\end{aligned}$$

Since $(f + g)(-x) \neq -(f + g)(x)$ and $(f + g)(-x) \neq (f + g)(x)$, the sum of f and g is neither even nor odd.